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ON STABILITY IN THE PRESENCE OF MULTIPLE RESONANCE OF ODD ORDER*

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The stability of the trivial solution of an autonomous system of ordinary differential equations is investigated in the critical case of n pairs of pure imaginary roots when odd-order multiple resonance is present. All possible cases of the presence of a third-order double resonance are examined for a canonic system. The stability problem for the relative equilibrium of a satellite on a circular orbit is analyzed as an example.

1. We consider the differential equation system

$$x^{\cdot} = Ax + X(x), \quad X(0) = 0, \quad x \in E_{2n} \tag{1.1}$$

where A is a constant square matrix having only pure imaginary and distinct eigenvalues $\pm i\omega_s(\omega_s > 0,\ s = 1, \ldots, n)$, $X(x)$ is a holomorphic vector-valued function whose expansion in powers of x begins with an mth-order form, m is an even number. We assume that system (1.1) has μ-ple internal resonance of order $(m+1)$, i.e., all possible resonance relations of the form

$$\langle \Omega, P_\nu \rangle = 0, \quad \nu = 1, \ldots, \mu \tag{1.2}$$

$$\Omega = (\omega_1, \ldots, \omega_q), \quad P_\nu = (p_{\nu 1}, \ldots, p_{\nu q}), \quad q \leqslant n$$

$$|P_\nu| = \sum_{j=1}^{q} |p_{\nu j}| = k, \quad k = m + 1$$

are fulfilled, where P_ν is an integral vector whose components do not contain a common factor. For definiteness we can take it that the first nonzero component of vector P_ν is positive.

The stability problem for the trivial solution of system (1.1) was investigated in /1–3/ in the presence of μ-ple resonance (1.2) satisfying the condition

$$p_{\nu j}^* = (-1)^{\alpha_\nu + \beta_j} p_{\nu j} \geqslant 0, \quad \nu = 1, \ldots, \mu, \quad j = 1, \ldots, q \tag{1.3}$$

for certain α_ν, β_j taking value 1 or 2. Below we examine this same problem without constraints (1.3). A special case of such a problem was analyzed earlier in /4/. With the aid of a special nonlinear transformation taking (1.2) into account the system (1.1) in polar coordinates $r_s, \varphi_s (s = 1, \ldots, n)$ can be reduced to the normal form /5/

$$r_j^{\cdot} = 2 \sum_{\nu=1}^{\mu} R_\nu Q_{\nu j}(\theta_\nu^*) + \Upsilon_j(r, \varphi), \qquad \theta_\nu^{*\cdot} = \sum_{i=1}^{\mu} \sum_{j=1}^{q} \frac{p_{\nu j}^* \operatorname{sign} p_{ij}^*}{r_j} R_i Q_{ij}'(\theta_i^*) + \Theta_{n+\nu}(r, \varphi) \tag{1.4}$$

$$r_\alpha^{\cdot} = \Upsilon_\alpha(r, \varphi), \quad r_\alpha \varphi_\alpha^{\cdot} = \omega_\alpha r_\alpha + \Theta_\alpha(r, \varphi), \qquad j = 1, \ldots, q, \quad \nu = 1, \ldots, \mu, \quad \alpha = q + 1, \ldots, n$$

$$R_\nu^2 = \prod_{l=1}^{q} r_l^{|p_{\nu l}^*|}, \quad \theta_\nu^* = \sum_{j=1}^{q} p_{\nu j}^* \varphi_j$$

$$Q_{\nu j}(\theta_\nu^*) = a_{\nu j} \cos \theta_\nu^* + b_{\nu j} \sin \theta_\nu^*, \quad Q_{\nu j}' = dQ_{\nu j}/d\theta_\nu^*$$

$$r = (r_1, \ldots, r_n), \quad \varphi = (\varphi_1, \ldots, \varphi_n), \quad \Theta_\alpha(r, \varphi) \sim O(\|r\|^{(k+1)/2})$$

$$\Theta_{n+\nu}(r, \varphi) = \sum_{j=1}^{q} \frac{1}{r_j} \Theta_{\nu j}(r, \varphi), \quad \Theta_{\nu j}(r, \varphi) \sim O(\|r\|^{(k+1)/2})$$

$$\Upsilon_s(r, \varphi) \sim O(\|r\|^{(k+1)/2}), \quad s = 1, \ldots, n, \qquad Q_{\nu j}(\theta_\nu^*) \equiv 0, \quad \text{if} \quad p_{\nu j}^* = 0$$

A corresponding model system is obtained from (1.4) when

$$\Upsilon_s(r, \varphi) = \Theta_\nu(r, \varphi) \equiv 0, \quad s = 1, \ldots, n, \quad \nu = 1, \ldots, \mu$$

It is assumed that

$$\sum_{j=1}^{q} |Q_{\nu j}| \not\equiv 0, \quad \nu = 1, \ldots, \mu$$

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We denote ($\delta_{\beta h}$ is the Kronecker symbol)

$$A_{\beta h}=\sum_{\nu=1}^{\mu} S_{\nu\beta}^{\circ}K_{\nu h}-2\delta_{\beta h}, \quad A_{\beta,\, n+\nu}=S_{\nu\beta}^{\circ\prime}$$

$$A_{n+\nu,\,\beta}=\sum_{i=1}^{\mu}(T_{\nu i}^{\circ\prime}K_{i\beta}-L_{\nu i\beta}), \quad A_{n+\nu,\, n+i}=-T_{\nu i}^{\circ}$$

$$K_{\nu\beta}=\frac{1}{2\sqrt{q-\beta}}\Big[\sum_{l=\beta+1}^{q}|p_{\nu l}^{*}|-(q-\beta)|p_{\nu\beta}^{*}|\Big]$$

$$L_{\nu i\beta}=\frac{R_i^{\circ}}{\sqrt{q-\beta}}\Big[\sum_{l=\beta+1}^{q}\frac{p_{\nu l}^{*}\operatorname{sign}p_{il}^{*}Q_{il}^{\circ\prime}}{k_l}-\frac{(q-\beta)\,p_{\nu\beta}^{*}\operatorname{sign}p_{i\beta}^{*}Q_{i\beta}^{\circ\prime}}{k_\beta}\Big]$$

$$S_{\nu\beta}(\theta_\nu^{*})=\frac{2R_\nu^{\circ}}{(q-\beta+1)\sqrt{q-\beta}}\Big[\sum_{l=\beta+1}^{q}\frac{Q_{\nu l}(\theta_\nu^{*})}{k_l}-\frac{(q-\beta)\,Q_{\nu\beta}(\theta_\nu^{*})}{k_\beta}\Big]$$

$$T_{\nu i}(\theta_i^{*})=R_i^{\circ}\sum_{j=1}^{q}\frac{p_{\nu j}^{*}\operatorname{sign}p_{ij}^{*}\,Q_{ij}(\theta_i^{*})}{k_j}$$

$$S_{\nu\beta}^{\circ}=S_{\nu\beta}(\theta_\nu^{*\circ}), \quad T_{\nu i}^{\circ}=T_{\nu i}(\theta_i^{*\circ})$$

$$\beta, h=1,\ldots,q-1, \quad \nu, i=1,\ldots,\mu$$

Theorem 1. If the model system has a particular solution of the growing ray type

$$r_j=k_jb(t), \quad b^{\cdot}=2b^{k/2}, \quad k_j>0, \quad j=1,\ldots,q, \qquad \theta_\nu^{*}=\theta_\nu^{*\circ}=\text{const}, \quad \nu=1,\ldots,\mu$$

and

$$\det\|A_{\upsilon\sigma}-l\delta_{\upsilon\sigma}\|\neq 0, \quad l=1,2,\ldots, \qquad \upsilon,\sigma=1,\ldots,n+\mu \ (\upsilon,\sigma\neq q,\ldots,n)$$

then the trivial solution of equations system (1.1) is Liapunov-unstable.

The proof is similar to that of the theorem in /3/. In the general case Theorem 1 does not help us to obtain constructive conditions for Liapunov-instability; however, in certain special cases (for instance, for a canonic system with third-order double resonance)the theorem does yield the constructive conditions.

2. We consider the canonic system

$$p_s^{\cdot}=-\frac{\partial H(p,q)}{\partial q_s}, \quad q_s^{\cdot}=\frac{\partial H(p,q)}{\partial p_s}, \quad p,q\in E_n, \quad s=1,\ldots,n \tag{2.1}$$

$$H(p,q)=\frac{1}{2}\sum_{s=1}^{n}(-1)^{\delta_s}(p_s^2+\omega_s^2q_s^2)+H_k+H_{k+1}+\ldots \tag{2.2}$$

where H_l is a homogeneous polynomial of degree l, δ_s takes the value 1 or 2, so that the quadratic form in (2.2) is indefinite; the linearized system does not have multiple eigenvalues and relations (1.2) are fulfilled. With the aid of a polynomial canonic transformation taking (1.2) into account the Hamiltonian (2.2) in polar canonic variables can be reduced to the normal form

$$\Gamma=\sum_{s=1}^{n}\lambda_sr_s+2\sum_{\nu=1}^{\mu}A_\nu R_\nu Q_\nu'(\theta_\nu^{*})+\Gamma^{*}(r,\varphi) \tag{2.3}$$

$$Q_\nu(\theta_\nu^{*})=a_\nu\cos\theta_\nu^{*}+b_\nu\sin\theta_\nu^{*}\equiv\sin\psi_\nu^{*}, \ a_\nu^2+b_\nu^2=1, \qquad Q_\nu'=dQ_\nu/d\theta_\nu^{*}, \quad \Gamma^{*}(r,\varphi)\sim O(\|r\|^{(k+1)/2})$$

$$\lambda_s=(-1)^{\delta_s}\omega_s, \quad p_{\nu j}^{*}=(-1)^{\delta_j}p_{\nu j}, \qquad \nu=1,\ldots,\mu, \quad j=1,\ldots,q, \quad s=1,\ldots,n$$

The model Hamiltonian is obtained from (2.3) when $\Gamma^{*}(r,\varphi)\equiv 0$. From Theorem 1 follows

Theorem 2. If the canonic system with the model Hamiltonian has a particular solution of the form

$$\begin{aligned} &r_j=k_jb(t), \ b^{\cdot}=2b^{k/2}, \ k_j>0, \ j=1,\ldots,q \\ &\psi_\xi^{*}=(\pi/2)\operatorname{sign}A_\xi, \quad \xi=1,\ldots,\mu_0 \\ &\psi_\eta^{*}=-(\pi/2)\operatorname{sign}A_\eta, \quad \eta=\mu_0+1,\ldots,\mu \ (0\leqslant\mu_0\leqslant\mu) \end{aligned} \tag{2.4}$$

and

$$\det\|A_{\upsilon\sigma}-l\delta_{\upsilon\sigma}\|\neq 0, \quad l=1,2,\ldots, \qquad \upsilon,\sigma=1,\ldots,n+\mu \ (\upsilon,\sigma\neq q,\ldots,n)$$

then the trivial solution of canonic system (2.1) is Liapunov-unstable.

Here

$$A_{\beta h}=\sum_{\nu=1}^{\mu}S_{\nu\beta}^{\circ}K_{\nu h}-2\delta_{\beta h}, \quad A_{\beta,\, n+\nu}=0$$

$$A_{n+\nu,\,\beta}=0,\quad A_{n+\nu,\,n+i}=(-1)^{\sigma_i+1}R_i^\circ|A_i|\sum_{j=1}^{q}\frac{p^*_{\nu j}|p^*_{ij}|}{k_j}$$

$$K_{\nu\beta}=\frac{1}{2\sqrt{q-\beta}}\Big[\sum_{l=\beta+1}^{q}|p^*_{\nu l}|-(q-\beta)|p^*_{\nu\beta}|\Big]$$

$$S^\circ_{\nu\beta}=\frac{2(-1)^{\sigma_\nu}R_\nu^\circ|A_\nu|}{(q-\beta+1)\sqrt{q-\beta}}\Big[\sum_{l=\beta+1}^{q}\frac{p^*_{\nu l}}{k_l}-\frac{(q-\beta)p^*_{\nu\beta}}{k_\beta}\Big]$$

$$\sigma_\xi=2,\quad \xi=1,\ldots,\mu_0,\quad \sigma_\eta=1,\quad \eta=\mu_0+1,\ldots,\mu$$

$$\beta,\ h=1,\ldots,q-1,\qquad \nu,\ i=1,\ldots,\mu$$

Let us assume that the canonic system (2.1) has a third-order double resonance and that among its resonance relations if only there is at least one strong (i.e., leading the zero solution of the model system to instability /1/). We investigate the stability question in this case by using the results in /2/ and in the present paper; to be precise, we study the following stability properties of the trivial solution of canonic system (2.1): instability in the second order because of the existence in the model system of particular solutions of the growing ray type of form

$$\begin{aligned}&r_u=k_ub(t),\ b^\cdot=2b^{k/2},\ k_u>0,\ u=1,\ldots,\bar q\\&r_v=0,\ v=\bar q+1,\ldots,q\quad(0<\bar q<q)\\&\psi_\xi{}^*=(\pi/2)\operatorname{sign}A_\xi,\ \xi=1,\ldots,\mu_0\\&\psi_\eta{}^*=-(\pi/2)\operatorname{sign}A_\eta,\ \eta=\mu_0+1,\ldots,\bar\mu\\&\psi_\zeta{}^*=\pm\pi/2,\ \zeta=\bar\mu+1,\ldots,\mu\quad(0\leqslant\mu_0\leqslant\bar\mu<\mu;\ \bar\mu>0)\end{aligned}\tag{2.5}$$

and of form (2.4); Liapunov-instability; stability in the second order with respect to a part of the variables.

We set up a table in which we enter the results of the stability investigations (sufficient conditions) in all possible cases of the presence of a third-order double resonance in system (2.1). In the Table 1 the double resonance (1.2) has been represented by the matrices

$$P^*=\|p_{\nu j}{}^*\|\ (\nu=1,\ 2;\ j=1,\ldots,q),\ A=|A_1/A_2|$$

in (2.3) A_l corresponds to the i-th resonance relation; $l=1,2,\ldots$; the asterisk in the table signifies that the stability property specified holds for $0<A<\infty$; the dash signifies that other investigation methods are needed to study the stability property specified. If in the third-order double resonance (1.2) all resonance relations are weak, then the trivial solution of canonic system (2.1) is stable in the second order (see /2/).

Example. We consider the stability problem for the relative equilibrium of a satellite on a circular orbit /6,7/. We investigate the stability for parameter values corresponding to an intersection of the resonance curves

$$\omega_3-\omega_2+\omega_1=0,\ \ \omega_3-2\omega_1=0\tag{2.6}$$

in the region wherein only the necessary stability conditions are fulfilled in the plane $\varepsilon=C/A$, $\delta=B/A$, where A,B,C are the satellite's principal central moments of inertia (see /7, 2/). Computer calculations showed that the double resonance (2.6) is realized at the point $\varepsilon_0=0.912686\ldots$, $\delta_0=0.835888\ldots$. By normalization the Hamiltonian of the problem being examined can be brought to form (2.3), where $A=1.492993\ldots<\sqrt{3}$. Since the double resonance (2.6) can be written as $\lambda_3-\lambda_2-\lambda_1=0$, $\lambda_3+2\lambda_1=0$, from the tabular results presented (see No.21) it follows that the relative equilibrium of the satellite on the circular orbit is Liapunov-unstable at the point $(\varepsilon_0;\ \delta_0)$.

Note. All the results obtained in /2/ remain valid for the multiple resonances (1.2) considered in the present paper. We remark that the conditions in Theorems 1.2 and 2.2 of /2/ are only sufficient and not necessary.

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Table 1

№	Form of the resonance	Instability in the second order: solution of form (2.5)	Instability in the second order: solution of form (2.4)	Liapunov instability	Stability in the second order with respect to a part of the variables
1	111000 000111	*	*	*	—
2	111000 00011—1	*	—	—	*
3	11100 10011	*	$A=1$	$A=1$	—
4	11100 100—1—1	*	$A=1$	$A=1$	—
5	11100 1001—1	*	—	—	*
6	11100 00021	*	*	*	—
7	11100 0002—1	*	—	—	*
8	11—100 00021	*	—	—	*
9	1110 1—101	—	$1<A<\infty$	$1<A<\infty$	—
10	1110 2001	*	$\sqrt{2}<A<\infty$	$\sqrt{2}<A<\infty$	—
11	1110 200—1	—	*	$0<A<\infty$ $A\neq\sqrt{\frac{2}{l+1}}$	—
12	1—1—10 2001	*	$0<A<\sqrt{2}$	$0<A<\sqrt{2}$ $A\neq\sqrt{\frac{2}{l+1}}$	—
13	11—10 2001	*	—	—	*
14	1110 1002	*	$A=2$	$A=2$	—
15	1110 100—2	*	$A=2$	$A=2$	—
16	1—1—10 1002	*	$A=2$	$A=2$	—
17	11—10 1002	*	—	—	*
18	2100 0021	*	*	*	—
19	2100 002—1	*	—	—	*
20	111 2—10	—	*	$0<A<\infty$ $A\neq\frac{2\sqrt{l+2}}{l+1}$	—
21	1—11 210	—	$0<A<\sqrt{3}$	$0<A<\sqrt{3}$	—
22	1—1—1 210	—	$0<A<1$	$0<A<1$	—
23	210 102	*	$0<A<\sqrt{2}$	$0<A<\sqrt{2}$	—
24	210 10—2	*	$\sqrt{2}<A<\infty$	$\sqrt{2}<A<\infty$ $A\neq\sqrt{2(l+1)}$	—
25	2—10 102	—	*	$0<A<\infty$ $A\neq\sqrt{2(l+1)}$	—

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